Energy of linear quasineutral electrostatic drift waves

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Certain kinds of nonlinear instabilities are related to the existence of negative-energy perturbations. In this paper, an exact energy expression for linear quasineutral electrostatic perturbations is derived within the framework of dissipationless multifluid theory that is valid for any geometry. Taking the mass formally as a tensor with, in general, different masses parallel and perpendicular to an ambient magnetic field allows one to treat in a convenient way different approximations such as the full dynamics and restriction to parallel dynamics or the completely adiabatic case. Application to slab configurations yields the result that the adiabatic approximation does not allow negative energy for perturbations which are perfectly localized at a mode resonant surface, whereas inclusion of the parallel dynamics does. This is in agreement with a recent numerical study of drift-wave turbulence within the framework of collisional two-fluid theory by B. Scott [Phys. Rev. Lett. **65**, 3289 (1990); Phys. Fluids B **4**, 2468 (1992)]. A dissipationless theory can be formulated in terms of a Lagrangian, from which the energy is immediately obtained. We start with the nonlinear theory. The theory is formulated via a Lagrangian which is written in terms of displacement vectors $\xi_\nu(\mathbf{x},t)$ such that all constraints are taken into account. The nonlinear energy is obtained from the Lagrangian by standard methods. The procedure used is the same as that developed in a forthcoming paper by Pfirsch and Sudan [Phys. Fluids B (to be published)] for ideal nonlinear magnetohydrodynamics theory. From the exact Lagrangian one obtains the Lagrangian of the linearized theory by simple expansion to second order in ξ_ν. This Lagrangian then yields the energy of the linearized theory. People working in this field have hitherto considered only (positive semidefinite) expressions for the nonlinear energy which do not immediately allow one to investigate the existence of linear negative-energy perturbations relating to nonlinear instability.

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I. INTRODUCTION

An impressive numerical study of collisional drift-wave turbulence was recently published by Scott [1,2] in which he demonstrated self-sustained turbulence of a linearly stable plasma slab resembling the plasma edge regions of tokamaks. His main results are that all features of nonlinear mode structure are determined by nonlinear processes, divesting linear stability criteria of their relevance to that structure, or its amplitude; contrary to the present common notion in tokamak physics that drift-wave turbulence cannot be the agent behind energy transport in tokamak-edge regions, many important features of experimentally observed tokamak-edge fluctuations were reproduced, particularly the amplitude ordering $e\tilde{\phi}/T > \tilde{n}/n > \tilde{T}/T$. The transport is found to be gyroreduced Bohm-diffusion-like. In Scott's study a certain threshold amplitude is needed. It can, however, even happen that the nonlinear instability occurs with arbitrarily small initial perturbations. This was shown for the first time in 1925 by Cherry [3]. He presented a simple example demonstrating that linear stability analysis in general will not be sufficient for finding out whether a system is stable or not with respect to small-amplitude perturbations. His example consisted of two nonlinearly coupled oscillators, one possessing positive energy, the other negative energy, and the frequency of one oscillator was twice that of the other, which means a third-order resonance. The exact two-parameter solution set he had found exhibited explosive instability after a finite time. Pfirsch [4] considered the corresponding three-oscillator case and found the complete solution of this problem. It shows that, except for a singular case, all initial conditions, especially those with arbitrarily small amplitudes, lead to explosive behavior. This is true of the resonant case. The nonresonant oscillators can sometimes also become explosively unstable, but the initial amplitudes must not be infinitesimally small. (Concerning negative-energy modes and nonlinear instabilities see also [5].)

It is very likely that the results obtained by Scott are understandable in terms of coupling between positive- and negative-energy perturbations and that the threshold amplitude he needed relates to a lack of resonance.

In order to find our whether this conjecture might be correct, it is necessary to determine the sign of the energy of drift-wave perturbations. People working in this field [6], especially Scott [1,2], have hitherto considered only (positive semidefinite) expressions for the nonlinear energy, which do not immediately allow one to investigate the existence of linear negative-energy perturbations relating to nonlinear instability [3–5]. In this paper we derive the exact expression for the energy of linear quasineutral electrostatic drift waves and evaluate it for situations corresponding to Scott's calculations. Contrary to Scott, we consider a dissipationless theory, which is the only kind allowing a unique definition of the energy of perturbations. Within such a theory the energy is a constant of the motion for any initial conditions. Whereas the nonlinear energy is just kinetic plus potential plus thermal energy, the energy of perturbations depends on constraints. In a multifluid quasineutral electrostatic theory, from which we start, such constraints are mass conservation and entropy conservation. The latter is violated if heat conduction, heat sources (e.g., Joule heating), and heat sinks play a role. Hence, the energy expressions obtained in this paper are, strictly speaking, only valid for situations where this is not the case or where these phenomena do not influence the entropy constraint. The latter is the case if the heat conduction is infinitely large such that the equilibrium temperature profiles $T_\nu(\mathbf{x})$ of the various particle species ν are independent of $\mathbf{x}$ and $\delta T_\nu = 0$. A vanishing temperature perturbation results in an entropy-conserving theory if one takes the adiabatic coefficients $\gamma_\nu = 1$. This is only possible, however, for the perturbations; the equilibrium energy would diverge. When we consider this case, we do it by putting the γ's equal to 1 only after having obtained the perturbed energy for general γ's.

In this context, we should like to mention that the present authors are also investigating the same kind of problems within the framework of Vlasov-Maxwell theory, starting from the energy expression obtained by Morrison and Pfirsch [7]. This theory, of course, automatically takes heat and momentum transport into account. They also intend to use the energy obtained by Pfirsch and Morrison [8] within the framework of Maxwell and drift-kinetic theory.

Scott had found that it is essential to take into account the dynamics parallel to the ambient magnetic field for electrons and ions, whereas the inertial effects connected with the perpendicular electron motion can be neglected. We can treat this in our investigation by expressing the mass of the particles as a tensor with different masses for the parallel and the perpendicular motion and choose $m_{e\perp} = 0$. Introducing the mass as a tensor is simply an artificial measure which facilitates the discussion of different approximations.

We shall explicitly discuss examples in slab geometry. We choose $T_i = \mathbf{v}_i = 0$, where $\mathbf{v}_i$ is the equilibrium flow velocity of the ions.

II. QUASINEUTRAL ELECTROSTATIC MULTIFLUID THEORY

A dissipationless theory can be formulated in terms of a Lagrangian from which the energy is immediately obtained. We start from the nonlinear theory. The theory is formulated in a holonomous way via a Lagrangian which is written in terms of displacement vectors $\boldsymbol{\xi}(\mathbf{x}, t)$ such that all constraints are taken into account. The procedure used is the same as that developed in a forthcoming paper by Pfirsch and Sudan [9] for ideal nonlinear magnetohydrodynamics (MHD) theory.

A. The Lagrangian of the nonlinear theory

In this section we prove that the Lagrangian (1) below yields the exact quasineutral multifluid theory. Quasineutrality means that there is no electric field energy term, and, since the magnetic field is even prescribed, there is no magnetic field energy term either; instead we have to use the usual coupling terms for charged particles in electric and magnetic fields. This results in

$$L = \sum_\nu \int d^3x \left\{ \frac{1}{2} n_\nu \mathbf{v}_\nu \cdot \underline{m}_\nu \cdot \mathbf{v}_\nu - \frac{p_\nu}{\gamma_\nu - 1} - e_\nu n_\nu \Phi + \frac{e_\nu}{c} n_\nu \mathbf{v}_\nu \cdot \mathbf{A} \right\} , \quad (1)$$

where

$$\underline{m}_\nu = m_{\nu\perp} \underline{I} + (m_{\nu\parallel} - m_{\nu\perp}) \mathbf{bb} , \quad I_{ik} = \delta_{ik} , \quad \mathbf{b} = \mathbf{B}/B . \quad (2)$$

The task is to prove that

$$\delta \int_{t_1}^{t_2} L \, dt = 0 \quad (3)$$

under the constraints of mass and entropy conservation yields the desired equations of motion. $\mathbf{A} = \mathbf{A}(\mathbf{x})$ is the fixed equilibrium vector potential and is not to be varied, but all other quantities are.

If one introduces displacements $\boldsymbol{\xi}(\mathbf{x}, t)$, the position of a plasma element at time t, which in the unperturbed system is at $\mathbf{x}$ at time t, is given by

$$\hat{\mathbf{x}}(t) = \mathbf{x} + \boldsymbol{\xi}(\mathbf{x}, t) \quad (4)$$

(perturbed quantities are denoted by a hat over the symbol; the index ν will be suppressed if it is not needed). The new velocity at the new position is

$$\hat{\mathbf{v}}(\hat{\mathbf{x}}, t) = \frac{d\hat{\mathbf{x}}}{dt} = \frac{d\mathbf{x}}{dt} + \frac{\partial \boldsymbol{\xi}}{\partial t} + \left(\frac{dx}{dt} \cdot \nabla \right) \boldsymbol{\xi} = \mathbf{v}(\mathbf{x}, t) + \dot{\boldsymbol{\xi}} + [\mathbf{v}(\mathbf{x}, t) \cdot \nabla] \boldsymbol{\xi} . \quad (5)$$

Equation (4) yields the perturbed volume element

$$d^3\hat{\mathbf{x}} = J(\mathbf{x}, t) d^3\mathbf{x} , \quad (6)$$

where $J(\mathbf{x}, t)$ is the Jacobian of the transformation (4). Mass conservation requires

$$\hat{n}(\hat{\mathbf{x}}, t) d^3\hat{\mathbf{x}} = n(\mathbf{x}, t) d^3\mathbf{x} , \quad (7)$$

while entropy conservation and Eqs. (6) and (7) yield

$$\hat{p}(\hat{\mathbf{x}}, t) = p(\mathbf{x}, t) \left[\frac{\hat{n}(\hat{\mathbf{x}}, t)}{n(\mathbf{x}, t)} \right]^\gamma = p(x, t) \frac{1}{[J(\mathbf{x}, t)]^\gamma} . \quad (8)$$

Therefore, the Lagrangian of the perturbed system, namely,

$$\hat{L} = \sum_\nu \int d^3\hat{x} \left\{ \tfrac{1}{2} \hat{n}_\nu(\hat{\mathbf{x}}, t) \hat{\mathbf{v}}_\nu(\hat{\mathbf{x}}, t) \cdot \underline{m}_\nu \cdot \hat{\mathbf{v}}_\nu(\hat{\mathbf{x}}, t) - \frac{\hat{p}_\nu(\hat{\mathbf{x}}, t)}{\gamma_\nu - 1} - e_\nu \hat{n}_\nu(\hat{\mathbf{x}}, t) \hat{\Phi}(\hat{\mathbf{x}}, t) + \frac{e_\nu}{c} \hat{n}_\nu(\hat{\mathbf{x}}, t) \hat{\mathbf{v}}_\nu(\hat{\mathbf{x}}, t) \cdot \mathbf{A}(\hat{\mathbf{x}}) \right\} , \quad (9)$$

can be written as

$$\hat{L}=\sum_\nu \int d^3x \left\{ \tfrac{1}{2}n_\nu(\mathbf{x},t)\{\mathbf{v}_\nu(\mathbf{x},t)+[\mathbf{v}_\nu(\mathbf{x},t)\cdot\nabla]\xi_\nu(\mathbf{x},t)+\dot{\xi}_\nu(\mathbf{x},t)\}\cdot\underline{m}_\nu\cdot\{\mathbf{v}_\nu(\mathbf{x},t)+[\mathbf{v}_\nu(\mathbf{x},t)\cdot\nabla]\xi_\nu(\mathbf{x},t)+\dot{\xi}_\nu(\mathbf{x},t)\} \right.$$
$$-\frac{1}{[\gamma_\nu-1]}\frac{p_\nu(\mathbf{x},t)}{[J_\nu(\mathbf{x},t)]^{\gamma_\nu-1}}-e_\nu n_\nu(\mathbf{x},t)\hat{\Phi}[\mathbf{x}+\xi_\nu(\mathbf{x},t),t]$$
$$\left. +\frac{e_\nu}{c}n_\nu(\mathbf{x},t)\{\mathbf{v}_\nu(\mathbf{x},t)+[\mathbf{v}_\nu(\mathbf{x},t)\cdot\nabla]\xi_\nu(\mathbf{x},t)+\dot{\xi}_\nu(\mathbf{x},t)\}\cdot\mathbf{A}(\mathbf{x}+\xi_\nu(\mathbf{x},t)) \right\} . \tag{10}$$

Taking into account Eqs. (B2) and (B3) derived in Appendix B and the fact that $\underline{m}_\nu$ is symmetric, and since terms which can be written as divergences integrate to zero for the perturbations which are considered here, one can write the perturbed Lagrangian as

$$\hat{L}=L+\sum_\nu \int d^3x \left\{ \frac{\partial}{\partial t}[n_\nu\xi_\nu\cdot\underline{m}_\nu\cdot\mathbf{v}_\nu]-n_\nu\xi_\nu\cdot\underline{m}_\nu\cdot\left[\frac{\partial\mathbf{v}_\nu}{\partial t}+(\mathbf{v}_\nu\cdot\nabla)\mathbf{v}_\nu\right] \right.$$
$$+\tfrac{1}{2}n_\nu[\dot{\xi}_\nu+(\mathbf{v}_\nu\cdot\nabla)\xi_\nu]\cdot\underline{m}_\nu\cdot[\dot{\xi}_\nu+(\mathbf{v}_\nu\cdot\nabla)\xi_\nu]-\frac{p_\nu}{[\gamma_\nu-1]}[J_\nu^{1-\gamma_\nu}-1]-e_\nu n_\nu[\hat{\Phi}(x+\xi_\nu)-\Phi(x)]$$
$$\left. +\frac{e_\nu}{c}n_\nu\left\{\mathbf{v}_\nu\cdot[\mathbf{A}(\mathbf{x}+\xi_\nu)-\mathbf{A}(x)]-\xi_\nu\cdot[(\mathbf{v}_\nu\cdot\nabla)\mathbf{A}(\mathbf{x}+\xi_\nu)]\right\}+\left[\frac{\partial}{\partial t}\left[\frac{e_\nu}{c}n_\nu\xi_\nu\right]\right]\cdot\mathbf{A}(\mathbf{x}+\xi_\nu)\right\} . \tag{11}$$

This equation is exact in the perturbations.

Hamilton's principle, Eq. (3), can now easily be evaluated since

$$\delta\int_{t_1}^{t_2}L\,dt=\int_{t_1}^{t_2}\delta L\,dt=\int_{t_1}^{t_2}L^{(1)}dt , \tag{12}$$

where $L^{(1)}$ is the first-order contribution of an expansion of the right-hand side (rhs) of Eq. (11) with respect to the perturbations and can be readily determined if one takes into account Eqs. (A11), (A14), and (A15). One obtains

$$L^{(1)}=\sum_\nu \int d^3x \left\{ \frac{\partial}{\partial t}[n_\nu\xi_\nu\cdot\underline{m}_\nu\cdot\mathbf{v}_\nu]-n_\nu\xi_\nu\cdot\underline{m}_\nu\cdot\left\{\frac{\partial\mathbf{v}_\nu}{\partial t}+(\mathbf{v}_\nu+\nabla)\mathbf{v}_\nu\right] \right.$$
$$\left. -(\xi_\nu\cdot\nabla p_\nu)-e_\nu n_\nu\delta^{(1)}\Phi-e_\nu n_\nu\xi_\nu\cdot\nabla\Phi+\frac{e_\nu}{c}n_\nu\xi_\nu\cdot[\mathbf{v}_\nu\times\mathbf{B}]+\frac{\partial}{\partial t}\left\{\frac{e_\nu}{c}n_\nu\xi_\nu\cdot\mathbf{A}\right]\right\} . \tag{13}$$

Here, use has been made of the fact that $\mathbf{A}(\mathbf{x})$ does not depend on t [in contrast to $\mathbf{A}(\mathbf{x}+\xi_\nu(\mathbf{x},t))$], and of the identity $\mathbf{v}_\nu\cdot[(\xi_\nu\cdot\nabla)\mathbf{A}]-\xi_\nu\cdot[(\mathbf{v}_\nu\cdot\nabla)\mathbf{A}]=\xi_\nu\cdot\mathbf{v}_\nu\times\mathbf{B}$. The first and last terms in the expression for $L^{(1)}$ do not contribute in Eq. (12), since they vanish after t integration.

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The requirement that the factor of $\delta^{(1)}\Phi$ must vanish leads to the quasineutrality condition

$$\sum_\nu e_\nu n_\nu=0 . \tag{14}$$

Vanishing of the factor of ξ_ν yields the correct nonlinear equations of motion:

$$n_\nu\underline{m}_\nu\left[\frac{\partial}{\partial t}+\mathbf{v}_\nu\cdot\nabla\right]\mathbf{v}_\nu=-\nabla p+e_\nu n_\nu\left[\mathbf{E}+\frac{1}{c}\mathbf{v}_\nu\times\mathbf{B}\right] , \tag{15}$$

with $\mathbf{E}=-\nabla\Phi$. It is thus proved that the Lagrangian, Eq. (1), is indeed the right one.

B. The Lagrangian of the linearized theory and the wave energy

This Lagrangian is given by $L^{(2)}$, the second-order contribution of the perturbations to the exact Lagrangian of the nonlinear theory, Eq. (11). It is determined in a way similar to $L^{(1)}$. One obtains

$$L^{(2)}=\sum_{\nu}\int d^3x\left\{\frac{1}{2}n_\nu[\dot{\xi}_\nu+(\mathbf{v}_\nu\cdot\nabla)\xi_\nu]\cdot\underline{m}_\nu\cdot[\dot{\xi}_\nu+(\mathbf{v}_\nu\cdot\nabla)\xi_\nu]\right.$$
$$-\frac{1}{2}(\nabla\cdot\xi_\nu)[(\xi_\nu\cdot\nabla p_\nu)+\gamma_\nu p_\nu\nabla\cdot\xi_\nu]+\frac{1}{2}\nabla p_\nu\cdot[(\xi_\nu\cdot\nabla)\xi_\nu]-e_\nu n_\nu\xi_\nu\cdot\nabla\delta^{(1)}\Phi-\frac{1}{2}e_\nu n_\nu\xi_\nu\xi_\nu:\nabla\nabla\Phi$$
$$\left.+\frac{1}{2}\frac{e_\nu}{c}n_\nu\mathbf{v}_\nu\cdot[\xi_\nu\xi_\nu:\nabla\nabla\mathbf{A}]-\frac{e_\nu}{c}n_\nu\xi_\nu\cdot\{(\mathbf{v}_\nu\cdot\nabla)[(\xi_\nu\cdot\nabla)\mathbf{A}]\}+\frac{e_\nu}{c}\left[\frac{\partial}{\partial t}[n_\nu\xi_\nu]\right][(\xi_\nu\cdot\nabla)\mathbf{A}]\right\}. \quad (16)$$

The terms involving Φ and the vector potential $\mathbf{A}$ in Eq. (16) can be transformed with the help of the results of Appendix B, Eqs. (B4), (B9), and (B10). Taking into account the equilibrium relations

$$n_\nu\underline{m}_\nu(\mathbf{v}_\nu\cdot\nabla)\mathbf{v}_\nu=-\nabla p_\nu+e_\nu n_\nu\left[\mathbf{E}+\frac{1}{c}\mathbf{v}_\nu\times\mathbf{B}\right], \quad (17)$$

one obtains the second-order Lagrangian for each particle species in the form

$$L_\nu^{(2)}=\int d^3x\left\{\tfrac{1}{2}n_\nu[\dot{\xi}_\nu+(\mathbf{v}_\nu\cdot\nabla)\xi_\nu]\cdot\underline{m}_\nu[\dot{\xi}_\nu+(v_\nu\cdot\nabla)\xi_\nu]\right.$$
$$-\tfrac{1}{2}(\nabla\cdot\xi_\nu)[(\xi_\nu\cdot\nabla p_\nu)+\gamma_\nu p_\nu\nabla\cdot\xi_\nu]-e_\nu n_\nu\xi_\nu\cdot\nabla\delta^{(1)}\Phi-\tfrac{1}{2}n_\nu[(\xi_\nu\cdot\nabla)\xi_\nu]\cdot\underline{m}_\nu\cdot(\mathbf{v}_\nu\cdot\nabla)\mathbf{v}_\nu$$
$$-\frac{1}{2n_\nu}\xi_\nu\cdot[\nabla p_\nu+n_\nu\underline{m}_\nu\cdot(\mathbf{v}_\nu\cdot\nabla)\mathbf{v}_\nu]\nabla\cdot(n_\nu\xi_\nu)+\frac{1}{2}\frac{e_\nu}{c}n_\nu\{[\dot{\xi}_\nu+(\mathbf{v}_\nu\cdot\nabla)\xi_\nu-(\xi_\nu\cdot\nabla)\mathbf{v}_\nu]\times\mathbf{B}\}\cdot\xi_\nu$$
$$\left.+\frac{1}{2}\frac{e_\nu}{c}\frac{\partial}{\partial t}[n_\nu\xi_\nu\cdot[(\xi_\nu\cdot\nabla)\mathbf{A}]]\right\}. \quad (18)$$

Applying Hamilton's principle to $L^{(2)}$ and varying with respect to $\delta^{(1)}\Phi$ yields the linearized quasineutrality condition

$$\sum_\nu e_\nu\nabla\cdot(n_\nu\xi_\nu)=0\,. \quad (19)$$

The linearized equations of motion for the ions and electrons can be obtained by applying Hamilton's principle to the second-order Lagrangian of each species, Eq. (18), and varying with respect to ξ_ν. An alternative method is to obtain the linearized equations directly from the nonlinear equations of motion, Eq. (15), with the perturbations $\delta^{(1)}\Phi$ and

$$\delta n_\nu=-\nabla\cdot(n_\nu\xi_\nu)\,, \quad (20)$$

$$\delta\mathbf{v}_\nu=\dot{\xi}_\nu+(\mathbf{v}_\nu\cdot\nabla)\xi_\nu-(\xi_\nu\cdot\nabla)\mathbf{v}_\nu \quad (21)$$

and

$$\delta p_\nu=-\xi_\nu\cdot\nabla p_\nu-\gamma_\nu p_\nu\nabla\cdot\xi_\nu\,. \quad (22)$$

With the Lagrangian density corresponding to $L_\nu^{(2)}$ being denoted by $\mathcal{L}_\nu^{(2)}$, the total energy density e is given by

$$e=\sum_\nu\left[\dot{\xi}_\nu\cdot\frac{\partial\mathcal{L}_\nu^{(2)}}{\partial\dot{\xi}_\nu}-\mathcal{L}_\nu^{(2)}\right]. \quad (23)$$

The expression for the total wave energy is therefore

$$\mathcal{E}=\tfrac{1}{2}\sum_\nu\int d^3x\left\{n_\nu\cdot\dot{\xi}_\nu\cdot\underline{m}_\nu\cdot\dot{\xi}_\nu-n_\nu[(\mathbf{v}_\nu\cdot\nabla)\xi_\nu]\cdot\underline{m}_\nu\cdot[(\mathbf{v}_\nu\cdot\nabla)\xi_\nu]+(\nabla\cdot\xi_\nu)[(\xi_\nu\cdot\nabla p_\nu)+\gamma_\nu p_\nu\nabla\cdot\xi_\nu]\right.$$
$$+n_\nu[(\xi_\nu\cdot\nabla)\xi_\nu]\cdot\underline{m}_\nu\cdot(\mathbf{v}_\nu\cdot\nabla)\mathbf{v}_\nu+\frac{1}{n_\nu}\xi_\nu\cdot[\nabla p_\nu+n_\nu\underline{m}_\nu\cdot(\mathbf{v}_\nu\cdot\nabla)\mathbf{v}_\nu]\nabla\cdot(n_\nu\xi_\nu)$$
$$\left.-\frac{e_\nu}{c}n_\nu\xi_\nu\cdot\{[(\mathbf{v}_\nu\cdot\nabla)\xi_\nu-(\xi_\nu\cdot\nabla)\mathbf{v}_\nu]\times\mathbf{B}\}\right\}. \quad (24)$$

Since $\mathcal{E}$ is a constant of the motion, it can be evaluated in terms of the initial conditions for ξ_ν and $\dot{\xi}_\nu$. In particular, one can find the minimum of $\mathcal{E}$ by varying $\xi_{\nu 0}$ and $\dot{\xi}_{\nu 0}$. The choice of the initial conditions is, however, not completely free. One constraint is the linearized quasineutrality condition, Eq. (19), restricting the possible ξ_ν's; but also the time derivative of the quasineutrality condition must be zero, which constrains the $\dot{\xi}_\nu$'s. Also the second time derivative must vanish. Inserting $\ddot{\xi}_\nu$ in it from the linearized equations of motion yields an equation for $\delta^{(1)}\Phi$ which, when used in the equations of motion, allows them to advance $\dot{\xi}_\nu$ in time for the next time step. If the tensors $\underline{m}_{\,\nu}$ are chosen such that certain components of $\ddot{\xi}_\nu$ do not appear in the equations of motion, further constraints exist. This will be discussed in more detail in the examples to come.

C. Adiabatic approximation

This approximation means

$$\underline{m}_{\,e}=0 . \tag{25}$$

We set

$$\underline{m}_{\,i}=m_i\underline{I} \tag{26}$$

and, like Scott [1,2], we choose in addition

$$p_i=0,\quad \mathbf{v}_i=\mathbf{0}\Longrightarrow\mathbf{E}=\mathbf{0} . \tag{27}$$

The wave energy is then given by

$$\mathcal{E}=\tfrac{1}{2}\int d^3x\left\{n_i m_i(\dot{\xi}_i)^2+(\nabla\cdot\xi_e)[(\xi_e\cdot\nabla p_e)+\gamma_e p_e\nabla\cdot\xi_e]\right.$$
$$\left.+\frac{1}{n_e}(\xi_e\cdot\nabla p_e)\nabla\cdot(n_e\xi_e)-e_e n_e\xi_e\{[(\mathbf{v}\cdot\nabla)\xi_e-(\xi_e\cdot\nabla)\mathbf{v}_e]\times\mathbf{B}\}\right\} . \tag{28}$$

If one introduces the current density

$$\mathbf{j}=e_e n_e\mathbf{v}_e \tag{29}$$

and makes use of the equilibrium equation

$$\nabla p_e=\frac{1}{c}\mathbf{j}\times\mathbf{B} \tag{30}$$

and of Eqs. (B14) and (B15), the expression for $\mathcal{E}$ can then be written as

$$\mathcal{E}=\tfrac{1}{2}\int d^3x\left[n_i m_i(\dot{\xi}_i)^2+(\nabla\cdot\xi_e)[(\xi_e\cdot\nabla p_e)+\gamma_e p_e\nabla\cdot\xi_e]+\frac{1}{c}(\xi_e\times\mathbf{j})\cdot\nabla\times(\xi_e\times\mathbf{B})\right] , \tag{31}$$

or else as

$$\mathcal{E}=\tfrac{1}{2}\int d^3x\left[n_i m_i(\dot{\xi}_i)^2+\gamma_e p_e(\nabla\cdot\xi_e)^2+\frac{1}{c}\xi_e\cdot\{\mathbf{j}\times[(\mathbf{B}\cdot\nabla)\xi_e-(\xi_e\cdot\nabla)B]\}\right] . \tag{32}$$

Equation (31) is reminiscent of the ideal MHD energy expression. Note that the last term enters the ideal MHD potential energy with the opposite sign. The reason for this difference is that there the magnetic field is not a fixed quantity but is determined self-consistently, which also leads to an additional term $(\delta\mathbf{B})^2/8\pi$.

The linearized equations of motion for the ions and the electrons can be obtained in the way explained in the previous subsection by applying Hamilton's principle to the second-order Lagrangian of each species, Eq. (18). However, for the adiabatic case with $\underline{m}_{\,e}=0$ and $p_i=0$, $\mathbf{v}_i=0$, it is easier to use the alternative method and obtain the linearized equations directly from the exact nonlinear equations of motion, Eq. (15). One obtains

$$m_i\ddot{\xi}_i=-e_i\nabla\delta^{(1)}\Phi+\frac{e_i}{c}\dot{\xi}_i\times\mathbf{B} \tag{33}$$

for the ions and

$$\nabla[\xi_e\cdot\nabla p_e+\gamma_e p_e\nabla\cdot\xi_e]-\frac{1}{n_e}[\nabla\cdot(n_e\xi_e)]\nabla p_e-e_e n_e\nabla\delta^{(1)}\Phi+\frac{e_e}{c}n_e(\dot{\xi}_e\times\mathbf{B})+\frac{e_e}{c}n_e[(\mathbf{v}_e\cdot\nabla)\xi_e-(\xi_e\cdot\nabla)\mathbf{v}_e]\times\mathbf{B}=\mathbf{0} \tag{34}$$

for the electrons.

If one takes into account that $\mathbf{B}\cdot\nabla n_e=0$ (which is a consequence of Eqs. (29) and (30) and of the adiabatic law $(\mathbf{v}_e\cdot\nabla)[p_e/n_e^{\gamma_\nu}]=0$), the parallel component of Eq. (34) yields

$$(\mathbf{B}\cdot\nabla)[\xi_e\cdot\nabla p_e+\gamma_e p_e\nabla\cdot\xi_e-e_e n_e\delta^{(1)}\Phi]=0 . \tag{35}$$

For a general perturbation ξ_e, this means, if one excludes nonlocal contributions from the ξ_ν's,

$$\delta^{(1)}\Phi = \frac{1}{e_e n_e}[\xi_e\cdot\nabla p_e + \gamma_e p_e \nabla\cdot\xi_e] = -\frac{\delta p_e}{e_e n_e} . \tag{36}$$

When this "adiabatic" relation is used in Eq. (34), this equation becomes

$$\frac{e_e}{c} n_e^2 B^2 \dot{\xi}_{e\perp} = [\nabla\cdot(n_e\xi_e)]\mathbf{B}\times\nabla p_e - [\xi_e\cdot\nabla p_e + \gamma_e p_e\nabla\cdot\xi_e]\mathbf{B}\times\nabla n_e - \frac{e_e}{c} n_e^2 \mathbf{B}\times\{[(\mathbf{v}_e\cdot\nabla)\xi_e - (\xi_e\cdot\nabla)\mathbf{v}_e]\times\mathbf{B}\} . \tag{37}$$

As an example, we consider a plasma slab dependent on x only; we choose

$$\gamma_e = 1 , \quad T_e = T = \text{const} , \quad \mathbf{v}_e = (0,0,v_e) = \text{const} . \tag{38}$$

We show that the minimum of $\mathcal{E}$ for this configuration is zero for perturbations which are perfectly localized at a mode resonant surface ($\mathbf{k}\cdot\mathbf{B}^{(0)}=0$). The equilibrium magnetic field

$$\mathbf{B} = B_y(x)\mathbf{e}_y + B_z\mathbf{e}_z \tag{39}$$

follows from

$$\nabla\times\mathbf{B} = \frac{4\pi}{c} e_e n_e v_e \mathbf{e}_z , \tag{40}$$

i.e., from

$$B_z' = 0 , \tag{41}$$

$$B_y' = \frac{4\pi}{c} e_e n_e(x) v_e = \frac{4\pi}{c} j_z . \tag{42}$$

In this case the equilibrium relation is

$$\nabla p_e = \frac{1}{c} e_e n_e \mathbf{v}_e\times\mathbf{B}$$

$$= -\frac{1}{c} e_e n_e(x) v_e B_y(x)\mathbf{e}_x . \tag{43}$$

The last term in Eq. (32) then yields

$$\frac{1}{c}\xi_e\cdot\{\mathbf{j}\times[(\mathbf{B}\cdot\nabla)\xi_e - (\xi_e\cdot\nabla)\mathbf{B}]\} = \frac{j_z}{c}\xi_e\cdot[(\mathbf{B}\cdot\nabla)(\mathbf{e}_z\times\xi_e) - (\xi_e\cdot\nabla)(\mathbf{e}_z\times\mathbf{B})]$$

$$= \frac{j_z}{c}\xi_e\cdot[-\mathbf{e}_x(\mathbf{B}\cdot\nabla\xi_{ey}) + \mathbf{e}_y(\mathbf{B}\cdot\nabla\xi_{ex}) + \mathbf{e}_x\xi_{ex}B_y'$$

$$= \frac{B_y'}{4\pi}[-\xi_{ex}(\mathbf{B}\cdot\nabla\xi_{ey}) + \xi_{ey}(\mathbf{B}\cdot\nabla\xi_{ex}) + B_y'\xi_{ex}^2] . \tag{44}$$

Since the equilibrium is independent of y and z, an appropriate ansatz for $\xi_e(\mathbf{x},t)$ is

$$\xi_e = \tfrac{1}{2}[\zeta_e(x,t)e^{i\mathbf{k}\cdot\mathbf{x}} + \zeta_e^*(x,t)e^{-i\mathbf{k}\cdot\mathbf{x}}] , \tag{45}$$

$$\mathbf{k} = k_y\mathbf{e}_y + k_z\mathbf{e}_z , \tag{46}$$

and correspondingly for ξ_i. This yields

$$\nabla\cdot\xi_e = \tfrac{1}{2}[(\zeta_{ex}' + i\mathbf{k}\cdot\zeta_e)e^{i\mathbf{k}\cdot\mathbf{x}} + (\zeta_{ex}^{*\prime} - i\mathbf{k}\cdot\zeta_e)e^{-i\mathbf{k}\cdot\mathbf{x}}] , \tag{47}$$

$$(\mathbf{B}\cdot\nabla)\xi_e = \frac{i}{2}(\mathbf{B}\cdot\mathbf{k})[\zeta_e(x,t)e^{i\mathbf{k}\cdot\mathbf{x}} - \zeta_e^*(x,t)e^{-i\mathbf{k}\cdot\mathbf{x}}] . \tag{48}$$

Inserting these results in Eq. (32) and subsequently integrating with respect to y and z over a periodicity surface s,

$$s = \frac{4\pi^2}{k_y k_z} , \tag{49}$$

yields the wave energy in the form

$$\mathcal{E}_{k_y k_z} = \frac{s}{8}\int dx \left\{ n_i m_i \dot{\zeta}_i\cdot\dot{\zeta}_i^* + p_e|\zeta_{ex}' + ik_y\zeta_{ey} + ik_z\zeta_{ez}|^2 + \frac{i}{2\pi}B_y'(\mathbf{B}\cdot\mathbf{k})[\zeta_{ex}\zeta_{ey}^* - \zeta_{ex}^*\zeta_{ey}] + \frac{1}{2\pi}(B_y')^2\zeta_{ex}\zeta_{ex}^* \right\} . \tag{50}$$

As mentioned in the preceding section, it is possible to discuss this expression in terms of initial conditions. It has already been emphasized that the initial conditions, however, are not completely arbitrary, since the constraint of quasineutrality, Eq. (19), is valid at all times. In terms of the complex displacements ζ_ν, the following relations must be satisfied:

$$\sum_\nu \left[e_\nu \frac{d}{dx}(n_\nu \zeta_{\nu x}) + i e_\nu n_\nu (\mathbf{k}\cdot\boldsymbol{\zeta}_\nu) \right] = 0 \tag{51}$$

and

$$\sum_\nu \left[e_\nu \frac{d}{dx}(n_\nu \dot{\zeta}_{\nu x}) + i e_\nu n_\nu (\mathbf{k}\cdot\dot{\boldsymbol{\zeta}}_\nu) \right] = 0 . \tag{52}$$

As far as the equations of motion, Eqs. (33) and (37), are concerned, $\dot{\boldsymbol{\zeta}}_{i0}$ and ζ_{i0} can be arbitrarily chosen. $\dot{\boldsymbol{\zeta}}_{e0}$, however is not completely free. This is because the equation of motion for the electrons is of first order since $m_e = 0$ in the adiabatic approximation. In the case treated here, with $\gamma_e = 1$, $T_e = \text{const}$, and $\mathbf{v}_e = v_e \mathbf{e}_z$, Eq. (37) reduces to

$$\dot{\boldsymbol{\xi}}_{e\perp} = -v_e \frac{\partial \boldsymbol{\xi}_{e\perp}}{\partial z} - \frac{c}{e_e}\frac{T_e}{p_e^2}\frac{1}{B^2}(\boldsymbol{\xi}_{e\perp}\cdot\nabla p_e)\mathbf{B}\times\nabla p_e . \tag{53}$$

This equation obviously does not contain either $\xi_\parallel$ or $\dot{\xi}_\parallel$. In terms of $\boldsymbol{\zeta}_e$, Eq. (53) means

$$\dot{\boldsymbol{\zeta}}_{e\perp} = -i k_z v_e \boldsymbol{\zeta}_{e\perp} - \frac{c}{e_e}\left[\frac{\mathbf{p}_e'}{p_e}\right]^2 \frac{T_e}{B^2}(\mathbf{B}\times\mathbf{e}_x)\zeta_{e\perp x} \tag{54}$$

and

$$\dot{\zeta}_{e\perp x} = -i k_z v_e \zeta_{ex} , \tag{55}$$

$$\begin{aligned} \dot{\zeta}_{e\perp y} &= -i k_z v_e \frac{B_z}{B^2}[B_z\zeta_{ey} - B_y\zeta_{ez}] - \frac{c}{e_e}\left[\frac{p_e'}{p_e}\right]^2 \frac{T_e}{B^2}B_z\zeta_{ex} \\ &= -i v_e \frac{B_z}{B^2}[(\mathbf{k}\cdot\mathbf{B})\zeta_{ey} - (\mathbf{k}\cdot\boldsymbol{\zeta}_e)B_y] - \frac{c}{e_e}\left[\frac{p_e'}{p_e}\right]^2 \frac{T_e}{B^2}B_z\zeta_{ex} , \end{aligned} \tag{56}$$

$$\begin{aligned} \dot{\zeta}_{e\perp z} &= i k_z v_e \frac{B_y}{B^2}[B_z\zeta_{ey} - B_y\zeta_{ez}] + \frac{c}{e_e}\left[\frac{p_e'}{p_e}\right]^2 \frac{T_e}{B^2}B_y\zeta_{ex} \\ &= i v_e \frac{B_y}{B^2}[(\mathbf{k}\cdot\mathbf{B})\zeta_{ey} - (\mathbf{k}\cdot\boldsymbol{\zeta}_e)B_y] - \frac{c}{e_e}\left[\frac{p_e'}{p_c}\right]^2 \frac{T_e}{B^2}B_y\zeta_{ex} . \end{aligned} \tag{57}$$

One could now minimize $\mathcal{E}_{k_y k_z}$, Eq. (50) in the usual way, introducing Lagrange multipliers to account for the constraint of quasineutrality, Eqs. (51) and (52). However, an easier and more straightforward way which yields the same results is to minimize $\mathcal{E}_{k_y k_z}$ *without* constraints and then show, *a posteriori*, that the constraints can be satisfied with the minimizing perturbations. This is now carried out.

In terms of initial conditions, the variation of $\mathcal{E}_{k_y k_z}$ with respect to $\dot{\boldsymbol{\zeta}}_{i0}^*$ yields

$$\dot{\boldsymbol{\zeta}}_{i0} = \mathbf{0} . \tag{58}$$

Variation of ζ_{ez0}^* yields $\zeta_{ex0}' = -i(\mathbf{k}\cdot\boldsymbol{\zeta}_{e0})$, i.e.,

$$i k_z \zeta_{ez0} = -\zeta_{ex0}' - i k_y \zeta_{ey0} , \tag{59}$$

which can always be satisfied with $k_z \neq 0$.

Variation of ζ_{ex0}^* yields

$$\zeta_{ex0} = \frac{i}{B_y'}(\mathbf{k}\cdot\mathbf{B})\zeta_{ey0} . \tag{60}$$

Inserting these results in Eq. (50), one obtains

$$(\mathcal{E}_{k_y k_z})_{\min} = -\frac{s}{16\pi}\int dx (\mathbf{k}\cdot\mathbf{B})^2 |\zeta_{ey0}|^2 . \tag{61}$$

With these results, the quasineutrality conditions. Eqs. (51) and (52) are, respectively,

$$\begin{aligned} \sum_\nu [e_\nu(n_\nu\zeta_{\nu x0})' + i e_\nu n_\nu(\mathbf{k}\cdot\boldsymbol{\zeta}_{\nu 0})] &= e_i(n_i\zeta_{ix0})' + i e_i n_i(\mathbf{k}\cdot\boldsymbol{\zeta}_{i0}) + e_e n_e'\zeta_{ex0} + e_e n_e \zeta_{ex0}' + i e_e n_e(k_y\zeta_{ey0} + k_z\zeta_{ez0}) \\ &= e_i(n_i\zeta_{ix0})' + i e_i n_i(\mathbf{k}\cdot\boldsymbol{\zeta}_{i0}) + e_e n_e'\zeta_{ex0} = 0 \end{aligned} \tag{62}$$

and

$$\sum_\nu [e_\nu(n_\nu\dot{\zeta}_{\nu x})' + i e_\nu n_\nu(\mathbf{k}\cdot\dot{\boldsymbol{\zeta}}_\nu)] = e_e n_e'\dot{\zeta}_{ex0} + e_e n_e \dot{\zeta}'_{ex0} + i e_e n_e(\mathbf{k}\cdot\mathbf{e}_B)\dot{\zeta}_{e\parallel 0} + i e_e n_e(\mathbf{k}\cdot\dot{\boldsymbol{\zeta}}_{e\perp 0}) = 0 . \tag{63}$$

Since ζ_i does not appear in the expression for $\mathcal{E}_{k_y k_z}$, ζ_{i0} can always be chosen in such a way that Eq. (62) is satisfied.

In Eq. (63), $\dot{\zeta}_{ex0}$ and $\dot{\boldsymbol{\zeta}}_{e\perp 0}$ are determined by Eqs. (55) and (54) as functions of ζ_{ex}, ζ_{ey}, and ζ_{ez}. $\dot{\zeta}_{e\parallel 0}$ is completely arbitrary and can always be chosen so as to satisfy Eq. (63) with the single exception of the points at which $(\mathbf{k}\cdot\mathbf{B}) = 0$. At these points, however $\dot{\zeta}_{ex0}$

$=\dot{\zeta}'_{ex0}=\dot{\zeta}_{e\perp y0}=\dot{\zeta}_{e\perp z0}=0$ follows from Eqs. (55)–(57), (59), (60), and (63) is also satisfied.

$(\mathcal{E}_{k_yk_z})_{\min}$, Eq. (61), is thus also the minimum of $\mathcal{E}_{k_yk_z}$ under the constraint of quasineutrality. In order to compare with the results of the nonlinear numerical calculations for drift-wave instabilities, with $\dot{\xi}_{i\perp}$ given by the drift approximation, we now consider localized perturbations, but still with $\dot{\zeta}_i$ arbitrary, which means $\dot{\zeta}_{i0}=0$ in order to minimize $\mathcal{E}_{k_yk_z}$, whereas in the usual theory of drift instabilities $\dot{\zeta}_i$ is approximated by the drift motion. We take $x=0$ as being a resonant surface and consider perturbations which are localized there and have $\mathbf{k}\cdot\mathbf{B}(0)\equiv\mathbf{k}\cdot\mathbf{B}^{(0)}=0$. Let ϵ_x be a small number which describes the localization width $2x_0$, $\epsilon_x\sim 2x_0$. If the perturbations were perfectly localized, then the energy *density* would exactly vanish, $[(\mathcal{E}_{k_yk_z})_{\min}]/[s\epsilon_x]=0$. But the perturbations have some extent across the resonant surface, and since $[\mathbf{k}\cdot\mathbf{B}(x)]^2=[k_yB_y'(0)]^2x^2+\cdots$, one only has $[(\mathcal{E}_{k_yk_z})_{\min}]/[s\epsilon_x]\approx 0$ because

$$(\mathcal{E}_{k_yk_z})_{\min}\approx-\frac{s}{16\pi}[k_yB_y'(0)]^2\int_{-x_0}^{x_0}dx\,x^2|\zeta_{ey0}|^2 \sim\epsilon_x^3\to 0\ . \tag{64}$$

The energy density $[(\mathcal{E}_{k_yk_z})_{\min}]/[s\epsilon_x]$ behaves as ϵ_x^2. This is in contrast to the case with included parallel electron dynamics. That case, considered in the next section, yields $[\mathcal{E}_{k_yk_z}]/[s\epsilon_x]$ finite.

Since Eqs. (61) and (64) were derived without the additional constraint imposed on $\dot{\zeta}_{i\perp}$ by the drift approximation, $(\mathcal{E}_{k_yk_z})_{\min}$ represents a lower bound for the $(\mathcal{E}_{k_yk_z})_{\min}$ which one would obtain by taking that constraint into account. Owing to the large ion mass, it seems reasonable to assume that also in the case of the drift approximation a large positive contribution from the ion inertia to the energy can only be avoided if $\dot{\zeta}_{i\perp}=0$. This can be shown if one considers that, in the drift approximation, $\dot{\xi}_{i\perp}$ is given by

$$\dot{\xi}_{i\perp}=\frac{c}{B^2}\mathbf{B}\times\nabla\delta^{(1)}\Phi \tag{65}$$

[this follows from Eq. (33) to leading order in an expansion in $(1/\Omega_i)\partial/\partial t$, where $\Omega_i=e_iB/m_ic$ is the ion gyrofrequency]. For the example considered here, with $\gamma_e=1$, $\delta^{(1)}\Phi$ is given by

$$\delta^{(1)}\Phi=\frac{1}{e_e}\frac{T_e}{n_e}\nabla\cdot[n_e\xi_e]\ . \tag{66}$$

In terms of complex displacements and potential, with

$$\delta^{(1)}\Phi=\tfrac{1}{2}[\phi(x,t)e^{i\mathbf{k}\cdot\mathbf{x}}+\phi^*(x,t)e^{-i\mathbf{k}\cdot\mathbf{x}}]\ , \tag{67}$$

Eqs. (65) and (66) mean that

$$\dot{\zeta}_{i\perp}=\frac{c}{B^2}\mathbf{B}\times[\phi'\mathbf{e}_x+i\phi\mathbf{k}] \tag{68}$$

and

$$\phi(x,t)=\frac{1}{e_e}\frac{T_e}{n_e}[(n_e\zeta_{ex})'+in_e\mathbf{k}\cdot\boldsymbol{\zeta}_e]\ , \tag{69}$$

respectively. The contribution of $\dot{\zeta}_{i\perp}$ to the energy is then

$$n_im_i\dot{\zeta}_{i\perp}\cdot\dot{\zeta}^*_{i\perp}=n_im_i\frac{c^2}{B^2}\left[\phi'\phi'^*+\left[\frac{\mathbf{B}}{B}\times\mathbf{k}\right]^2\phi\phi^*\right]\ . \tag{70}$$

The energy in the drift approximation is therefore given by

$$\mathcal{E}_{k_yk_z}=\frac{s}{8}\int dx\left\{n_im_i\dot{\zeta}_{i\parallel}\dot{\zeta}^*_{i\parallel}+n_im_i\frac{c^2}{B^2}\left[\phi'\phi'^*+\left[\frac{\mathbf{B}}{B}\times\mathbf{k}\right]^2\phi\phi^*\right]\right.$$
$$\left.+p_e|\zeta'_{ex}+ik_y\zeta_{ey}+ik_z\zeta_{ez}|^2+\frac{i}{2\pi}B_y'(\mathbf{B}\cdot\mathbf{k})[\zeta_{ex}\zeta^*_{ey}-\zeta^*_{ex}\zeta_{ey}]+\frac{1}{2\pi}(B_y')^2\zeta_{ex}\zeta^*_{ex}\right\}\ . \tag{71}$$

The term $k_z\zeta_{ez}$ can be eliminated from this equation if one takes into account relation (69), which leads to

$$\zeta'_{ex}+ik_y\zeta_{ey}+ik_z\zeta_{ez}=\frac{e_e}{T_e}\phi-\frac{n_e'}{n_e}\zeta_{ex}\ . \tag{72}$$

Variation of $\mathcal{E}_{k_yk_z}$ with respect to $\dot{\zeta}^*_{i\parallel 0}$ yields

$$\dot{\zeta}_{i\parallel 0}=0\ . \tag{73}$$

Variation with respect to ϕ_0^* leads to

$$\phi_0=0 \tag{74}$$

in the limit of large ion mass, which means that

$$T_e\gg\frac{1}{m_i}\left[\frac{e_iB}{ck}\right]^2=m_i\frac{\Omega_i^2}{k^2}\sim\frac{T_i}{k^2R_{gi}^2}\ , \tag{75}$$

where R_{gi} is the ion gyroradius.

As a consequence of Eqs. (66), (67), and (74), one now has

$$\nabla\cdot[n_e\xi_{e0}]=\mathbf{0}\ , \tag{76}$$

instead of $\nabla\cdot\xi_{e0}=\mathbf{0}$, which one obtained with $\dot{\zeta}_{i\perp}$ completely free.

Subsequent variation of $\mathcal{E}_{k_yk_z}$ with respect to ζ^*_{ex0} then yields, if the equilibrium relation, viz. $n_e'=p_e'/T_e=-(1/4\pi)(B_yB_y'/T_e)$ is taken into account,

$$\zeta_{ex0}=\frac{i}{2\pi}\frac{B_y'}{\left[\frac{(B_y')^2}{2\pi}+\frac{n_e'^2}{n_e}T_e\right]}(\mathbf{k}\cdot\mathbf{B})\zeta_{ey0}$$

$$=\frac{i}{B_y'}\frac{1}{\left[1+\frac{1}{8\pi}\frac{B_y^2}{p_e}\right]}(\mathbf{k}\cdot\mathbf{B})\zeta_{ey0} \quad (77)$$

[in contrast to Eq. (60)] and

$$(\mathcal{E}_{k_yk_z})_{\text{drift min}}=-\frac{s}{16\pi}\int dx\frac{1}{\left[1+\frac{1}{8\pi}\frac{B_y^2}{p_e}\right]^2}\times(\mathbf{k}\cdot\mathbf{B})^2|\zeta_{ey0}|^2 \quad (78)$$

[in contrast to Eq. (61)].

As in Eq. (62), it can be shown that the quasineutrality condition, Eq. (51) can be satisfied by choosing the free ζ_{i0}. It is also straightforward to show that the quasineutrality condition, Eq. (52) can be satisfied by choosing the free $\zeta_{e\|0}$ for $(\mathbf{k}\cdot\mathbf{B}\neq0)$, and that it is trivially satisfied for $\mathbf{k}\cdot\mathbf{B}=0$.

The dependence of $(\mathcal{E}_{k_yk_z})_{\text{drift min}}$ on $\mathbf{k}\cdot\mathbf{B}$ is the same as in Eq. (61).

D. Parallel dynamics

The last subsection has shown that the thermal energy alone does not lead to negative energy for perturbations which are perfectly localized at a mode resonant surface, where $\mathbf{k}\cdot\mathbf{B}^{(0)}=0$. We therefore now take the parallel electron dynamics into account and consider the same equilibrium of the preceding section, $\gamma_e=1$, $T=T_e=\text{const}$, $\mathbf{v}_e=(0,0,v_e)=\text{const}$, $p_i=\mathbf{v}_i=0$.

The masses are now

$$\underline{m}_i=m_i\underline{I}\ ,\quad m_{e\perp}=0\ ,\quad m_{e\|}=m_e\ . \quad (79)$$

In this case the energy, Eq. (24), is

$$\mathcal{E}=\tfrac{1}{2}\int d^3x\left\{n_im_i(\dot{\xi}_i)^2+n_em_e\dot{\xi}_{e\|}^2-n_em_e(\mathbf{v}_e\cdot\nabla\xi_{e\|})^2+p_e(\nabla\cdot\xi_e)^2+\frac{1}{c}\xi_e\cdot\{\mathbf{j}\times[(\mathbf{B}\cdot\nabla)\xi_e-(\xi_e\cdot\nabla)\mathbf{B}]\}\right\}. \quad (80)$$

Proceeding in a way similar to that in the previous section, one obtains the energy in the form

$$\mathcal{E}_{k_yk_z}=\frac{s}{8}\int dx\left\{n_im_i\dot{\zeta}_i\cdot\dot{\zeta}_i^*+n_em_e\dot{\zeta}_{e\|}\dot{\zeta}_{e\|}^*-n_em_e(\mathbf{k}\cdot\mathbf{v}_e)^2\zeta_{e\|}\zeta_{e\|}^*+p_e|\zeta_{ex}'+i\mathbf{k}\cdot\zeta_e|^2+\frac{i}{2\pi}(\mathbf{k}\cdot\mathbf{B})B_y'[\zeta_{ex}\zeta_{ey}^*-\zeta_{ex}^*\zeta_{ey}]+\frac{1}{2\pi}(B_y')^2\zeta_{ex}\zeta_{ex}^*\right\}. \quad (81)$$

One can now show that both positive- and negative-energy perturbations are possible, and that quasineutrality is satisfied. A simple discussion is again possible in terms of initial conditions. We choose

$$\zeta_{i0}=0\ ,\quad \dot{\zeta}_{i0}=0\ \ (\text{correspondingly}\ ,\ \phi_0=0)\ , \quad (82)$$

$$\zeta_{e0}=a_0(x)\mathbf{e}_x\times\mathbf{k}\ , \quad (83)$$

$$\dot{\zeta}_{e0}=\dot{a}_0(x)\mathbf{e}_x\times\mathbf{k}\ . \quad (84)$$

Therefore one gets

$$\zeta_{ex0}=\dot{\zeta}_{ex0}=0\ ,\quad \mathbf{k}\cdot\zeta_{e0}=\mathbf{k}\cdot\dot{\zeta}_{e0}=0\ , \quad (85)$$

and

$$\nabla\cdot\zeta_{e0}=\zeta_{ex0}'=0\ ,\quad (n_e\zeta_{ex0})'+in_e\mathbf{k}\cdot\zeta_{e0}=0\ . \quad (86)$$

The conditions for quasineutrality, Eqs. (51) and (52), are obviously satisfied.

The expression for the energy reduces to

$$\mathcal{E}_{k_yk_z}=\frac{s}{8}\int dxn_em_e\{|\dot{\zeta}_{e\|0}|^2-(k_zv_e)^2|\zeta_{e\|0}|^2\}\ . \quad (87)$$

Since $\mathbf{k}\times\mathbf{B}=(k_yB_z-k_zB_y)\mathbf{e}_x$, one then obtains

$$|\zeta_{e\|0}|^2=\frac{|a_0|^2}{B^2}[\mathbf{e}_x\cdot(\mathbf{k}\times\mathbf{B})]^2=\frac{|a_0|^2}{B^2}(k_yB_z-k_zB_y)^2\ , \quad (88)$$

$$|\dot{\zeta}_{e\|0}|^2=\frac{|\dot{a}_0|^2}{B^2}[\mathbf{e}_x\cdot(\mathbf{k}\times\mathbf{B})]^2=\frac{|\dot{a}_0|^2}{B^2}(k_yB_z-k_zB_y)^2 \quad (89)$$

and

$$\mathcal{E}_{k_yk_z}=\frac{s}{8}\int dx\ n_em_e\left[\mathbf{k}\times\frac{\mathbf{B}}{B}\right]^2\{|\dot{a}_0|^2-(k_zv_e)^2|a_0|^2\}\ . \quad (90)$$

At the mode resonant surface, $[\mathbf{k}\times\mathbf{B}^{(0)}]^2=[kB^{(0)}]^2-(\mathbf{k}\cdot\mathbf{B}^{(0)})^2=[kB^{(0)}]^2$. For localized perturbations with localization width $2x_0\sim\epsilon_x$, the energy density $\mathcal{E}_{k_yk_z}/s\epsilon_x$ remains finite for vanishing ϵ_x

$$\frac{\mathcal{E}_{k_yk_z}}{s\epsilon_x}\approx\frac{1}{16x_0}k^2\int_{-x_0}^{x^0}dx\ n_em_e\{|\dot{a}_0|^2-(k_zv_e)^2|a_0|^2\}\ , \quad (91)$$

in contrast to the adiabatic case, where it vanishes as ϵ_x^2. Since $\dot{a}_0$ and a_0 can be presribed independently of each other, one can have both positive- and negative-energy perturbations. This situation is similar to the Cherry oscillator case. The configuration considered should therefore allow nonlinear instabilities. A possibly necessary threshold amplitude will depend on the degree of resonance present in the initial conditions.

As concerns negative-energy modes, which are necessarily stable, their frequencies must obviously satisfy

$$\left[\frac{\omega}{k_z v_e}\right]^2 < 1 \; . \tag{92}$$

III. SUMMARY

An exact energy expression for linear quasineutral electrostatic perturbations has been derived within the framework of dissipationless multifluid theory that is valid for any geometry. Taking the mass as a tensor with, in general, different masses parallel and perpendicular to an ambient magnetic field allowed us to treat the full dynamics, but also to restrict consideration to parallel dynamics only or to the completely adiabatic case. Application to slab configurations yields results in agreement with Scott's [1,2] numerical study within the framework of collisional two-fluid theory. Arguments were given that a comparison with such a theory is reasonable. The result is that in plane geometry the adiabatic approximation does not allow negative energy for perturbations which are perfectly localized at a mode resonant surface, whereas inclusion of the parallel dynamics does. This might, of course, be different with other kinds of configurations, but parallel dynamics, as pointed out by Scott, should always play an essential role.

APPENDIX A: PERTURBED QUANTITIES

In this appendix, the quantities $J_\nu(\mathbf{x},t)$, $\hat{\Phi}(\mathbf{x}+\xi,t)$, and $\mathbf{A}(\mathbf{x}+\xi)$, which appear in the perturbed Lagrangian, Eq. (10), are calculated by the same method as in Ref. [9] to second order in the perturbations. For this purpose, we use the expressions for the perturbed position and volume element, Eqs. (4) and (6), respectively, and take into account the constraints imposed by mass and entropy conservation, Eqs. (7) and (8), respectively.

It is convenient to introduce the *normalized density* $\hat{N}$ by defining

$$\hat{N}(\hat{\mathbf{x}},t)=\frac{\hat{n}(\hat{\mathbf{x}},t)}{n(\mathbf{x},t)} \; . \tag{A1}$$

Expansion of $\hat{N}(\hat{\mathbf{x}},t)$ with respect to $\xi(\mathbf{x},t)$ in the argument yields

$$\hat{N}(\hat{\mathbf{x}},t)=\hat{N}(\mathbf{x},t)+\xi\cdot\nabla\hat{N}(\mathbf{x},t)+\tfrac{1}{2}\xi\xi{:}\nabla\nabla\hat{N}(\mathbf{x},t)+\cdots \; , \tag{A2}$$

with $\xi\xi{:}\nabla\nabla=\xi_i\xi_j(\partial^2/\partial x_i\partial x_j)$ in Cartesian coordinates. The perturbed normalized density at the original position can be expressed in terms of the displacements and is given by

$$\hat{N}(\mathbf{x},t)=1+\delta^{(1)}N(\mathbf{x},t)+\tfrac{1}{2}\delta^{(2)}N(\mathbf{x},t)+\cdots \; , \tag{A3}$$

with $\delta^{(1)}N$ given by the well-known expression

$$\delta^{(1)}N=-\nabla\cdot[N\xi] \; , \qquad \delta^{(1)}N_{N=1}=-\nabla\cdot\xi \tag{A4}$$

and $\delta^{(2)}N$ by

$$\delta^{(2)}N=\delta^{(1)}[\delta^{(1)}N] \; . \tag{A5}$$

$\delta^{(1)}\xi$ is defined as

$$\delta^{(1)}\xi=\hat{\xi}(\mathbf{x},t)-\xi(\mathbf{x},t) \; , \tag{A6}$$

which , since the new displacement at $\mathbf{x}$ is the old displacement at $\mathbf{x}-\xi(\mathbf{x},t)$, i.e.,

$$\hat{\xi}(\mathbf{x},t)=\xi(\mathbf{x}-\xi(x,t),t) \; , \tag{A7}$$

yields

$$\delta^{(1)}\xi(\mathbf{x},t)=-[\xi(\mathbf{x},t)\cdot\nabla]\xi(\mathbf{x},t) \; . \tag{A8}$$

It is now easy to calculate the Jacobian $J(\mathbf{x},t)$, which appears in the expression for the perturbed Lagrangian $\hat{L}$. It follows from Eqs. (6), (7), and (A1) that

$$J(\mathbf{x},t)=\frac{d^3\hat{\mathbf{x}}}{d^3\mathbf{x}}=\frac{n(\mathbf{x},t)}{\hat{n}(\hat{x},t)}=\hat{N}(\hat{\mathbf{x}},t)^{-1} \; . \tag{A9}$$

Up to second order in the displacements, this yields

$$J=1+\nabla\cdot\xi+\tfrac{1}{2}\{\nabla\cdot[\xi(\nabla\cdot\xi)]-\nabla\cdot[(\xi\cdot\nabla)\xi]\} \tag{A10}$$

and

$$J^{1-\gamma}=1-(\gamma-1)\nabla\cdot\xi+\tfrac{1}{2}(\gamma-1)\{-\nabla\cdot[\xi(\nabla\cdot\xi)]+\nabla\cdot[(\xi\cdot\nabla)\xi]\}+\tfrac{1}{2}\gamma(\gamma-1)(\nabla\cdot\xi)^2 \; . \tag{A11}$$

The perturbed electric potential at the new position, $\hat{\Phi}(\mathbf{x}+\xi,t)$, can be similarly calculated. Up to second order, one has

$$\hat{\Phi}(\mathbf{x}+\xi,t)=\hat{\Phi}(\mathbf{x},t)+(\xi\cdot\nabla)\hat{\Phi}(\mathbf{x},t)+\tfrac{1}{2}\xi\xi{:}\nabla\nabla\hat{\Phi}(\mathbf{x},t) \tag{A12}$$

and

$$\hat{\Phi}(\mathbf{x},t)=\Phi(\mathbf{x})+\delta^{(1)}\Phi+\tfrac{1}{2}\delta^{(2)}\Phi \; . \tag{A13}$$

Therefore one gets

$$\hat{\Phi}(\mathbf{x}+\xi,t)=\Phi(\mathbf{x})+\delta^{(1)}\Phi(\mathbf{x})+(\xi\cdot\nabla)\Phi(\mathbf{x})+(\xi\cdot\nabla)\delta^{(1)}\Phi(\mathbf{x})+\tfrac{1}{2}\delta^{(2)}\Phi(\mathbf{x})+\tfrac{1}{2}\xi\xi:\nabla\nabla\Phi(x)\ . \tag{A14}$$

The vector potential at the new position is (also up to second order)

$$\mathbf{A}(\mathbf{x}+\xi)=\mathbf{A}(\mathbf{x})+(\xi\cdot\nabla)\mathbf{A}(\mathbf{x})+\tfrac{1}{2}\xi\xi:\nabla\nabla\,\mathbf{A}(\mathbf{x})\ . \tag{A15}$$

There are no $\delta\,\mathbf{A}$ terms since $\mathbf{A}$ is prescribed (fixed). Also, $\mathbf{A}$ does not depend explicitly on time.

APPENDIX B: USEFUL RELATIONS TO TRANSFORM THE PERTURBED LAGRANGIAN AND THE WAVE ENERGY

The expression for the perturbed Lagrangian, Eq. (10), can be put in a convenient form by means of the relations derived in this appendix.

Taking into account the equation of continuity, i.e.,

$$\dot{n}_\nu+\nabla\cdot(n_\nu\mathbf{v}_\nu)=0\ , \tag{B1}$$

it is easy to verify the relations

$$\frac{1}{2}n_\nu\mathbf{v}_\nu\cdot\underline{m}_\nu\cdot\left[\frac{\partial}{\partial t}+(\mathbf{v}_\nu\cdot\nabla)\right]\xi_\nu=\frac{\partial}{\partial t}\left[\tfrac{1}{2}n_\nu\mathbf{v}_\nu\cdot\underline{m}_\nu\cdot\xi_\nu\right]+\nabla\{\tfrac{1}{2}n_\nu[\mathbf{v}_\nu\cdot\underline{m}_\nu\cdot\xi_\nu]\mathbf{v}_\nu\}-\frac{1}{2}n_\nu\left[\frac{\partial\mathbf{v}_\nu}{\partial t}+(\mathbf{v}_\nu\cdot\nabla)\mathbf{v}_\nu\right]\cdot\underline{m}_\nu\cdot\xi_\nu \tag{B2}$$

and

$$\frac{e_\nu}{c}n_\nu\left[\frac{\partial\xi_\nu}{\partial t}+(\mathbf{v}_\nu\cdot\nabla)\xi_\nu\right]\cdot\mathbf{A}(\mathbf{x}+\xi_\nu)=\frac{\partial}{\partial t}\left[\frac{e_\nu}{c}n_\nu\xi_\nu\right]\cdot\mathbf{A}(\mathbf{x}+\xi_\nu)$$
$$+\nabla\cdot\left[\frac{e_\nu}{c}[\xi_\nu\cdot\mathbf{A}(\mathbf{x}+\xi_\nu)]n_\nu\mathbf{v}_\nu\right]-\frac{e_\nu}{c}n_\nu\xi_\nu\cdot[(\mathbf{v}_\nu\cdot\nabla)\mathbf{A}(\mathbf{x}+\xi_\nu)]\ . \tag{B3}$$

Further relations are useful to transform the Lagrangian of the linearized theory, Eq. (16):

$$\begin{aligned}\tfrac{1}{2}e_\nu n_\nu\xi_\nu\xi_\nu:\nabla\nabla\Phi&=\tfrac{1}{2}e_\nu n_\nu\xi_{\nu i}\xi_{\nu j}\frac{\partial^2\Phi}{\partial x_i\partial x_j}\\&=\frac{\partial}{\partial x_i}\left[\frac{1}{2}e_\nu n_\nu\xi_{\nu i}\xi_{\nu j}\frac{\partial\Phi}{\partial x_j}\right]-\frac{1}{2}e_\nu\left[\frac{\partial}{\partial x_i}(n_\nu\xi_{\nu i})\right]\xi_{\nu j}\frac{\partial\Phi}{\partial x_j}-\tfrac{1}{2}e_\nu n_\nu\xi_{\nu i}\frac{\partial\xi_{\nu j}}{\partial x_i}\frac{\partial\Phi}{\partial x_j}\\&=\nabla\cdot[\tfrac{1}{2}e_\nu n_\nu(\xi_\nu\cdot\nabla\Phi)\xi_\nu]-\tfrac{1}{2}(\xi_\nu\cdot\nabla\Phi)\nabla\cdot(n_\nu\xi_\nu)-\tfrac{1}{2}e_\nu n_\nu[(\xi_\nu\cdot\nabla)\xi_\nu]\cdot\nabla\Phi\ ,\end{aligned} \tag{B4}$$

$$\begin{aligned}\nabla\cdot\{[\xi_\nu\cdot(\mathbf{v}_\nu\times\mathbf{B})]n_\nu\xi_\nu\}&=[\xi_\nu\cdot(\mathbf{v}_\nu\times\mathbf{B})]\nabla\cdot(n_\nu\xi_\nu)+n_\nu(\xi_\nu\cdot\nabla)[\xi_\nu\cdot(\mathbf{v}_\nu\times\mathbf{B})]\\&=[\xi_\nu\cdot(\mathbf{v}_\nu\times\mathbf{B})]\nabla\cdot(n_\nu\xi_\nu)+n_\nu[(\xi\cdot\nabla)\xi_\nu]\cdot(\mathbf{v}_\nu\times\mathbf{B})\\&\quad+n_\nu\xi_\nu\cdot\{[(\xi_\nu\cdot\nabla)\mathbf{v}_\nu]\times\mathbf{B}\}+n_\nu\xi_\nu\cdot\{\mathbf{v}_\nu\times[(\xi_\nu\cdot\nabla)\mathbf{B}]\}\ ,\end{aligned} \tag{B5}$$

$$\begin{aligned}\nabla\cdot\{[\xi_\nu\cdot[(\xi_\nu\cdot\nabla)\mathbf{A}]\}n_\nu\mathbf{v}_\nu]&=\{\xi_\nu\cdot[(\xi_\nu\cdot\nabla)\mathbf{A}]\}\nabla\cdot(n_\nu\mathbf{v}_\nu)+n_\nu(\mathbf{v}_\nu\cdot\nabla)\{\xi_\nu\cdot[(\xi_\nu\cdot\nabla)\mathbf{A}\}\\&=\{\xi_\nu\cdot[(\xi_\nu\cdot\nabla)\mathbf{A}]\}\nabla\cdot(n_\nu\mathbf{v}_\nu)+n_\nu[(\mathbf{v}_\nu\cdot\nabla)\xi_\nu]\cdot[(\xi_\nu\cdot\nabla)\mathbf{A}]+n_\nu\xi_\nu\cdot\{(\mathbf{v}_\nu\cdot\nabla)[(\xi_\nu\cdot\nabla)\mathbf{A}]\}\ ,\end{aligned} \tag{B6}$$

$$\begin{aligned}n_\nu\xi_\nu\cdot\{\mathbf{v}_\nu\times[(\xi_\nu\cdot\nabla)\mathbf{B}]\}&=n_\nu\xi_\nu\xi_{\nu i}\cdot\left[\mathbf{v}_\nu\times\frac{\partial\mathbf{B}}{\partial x_i}\right]\\&=n_\nu\xi_\nu\xi_{\nu i}\cdot\left[\mathbf{v}_\nu\times\nabla\times\frac{\partial\mathbf{A}}{\partial x_i}\right]\\&=n_\nu\xi_\nu\xi_{\nu i}\cdot\left[\nabla\left[\mathbf{v}_{\nu,\mathrm{const.}}\cdot\frac{\partial\mathbf{A}}{\partial x_i}\right]-(\mathbf{v}_\nu\cdot\nabla)\frac{\partial\mathbf{A}}{\partial x_i}\right]\\&=n_\nu\xi_{\nu j}\xi_{\nu i}\mathbf{v}_\nu\cdot\frac{\partial^2\mathbf{A}}{\partial x_j\partial x_i}-n_\nu\xi_\nu\cdot\left[\xi_{\nu i}(\mathbf{v}_\nu\cdot\nabla)\frac{\partial\mathbf{A}}{\partial x_i}\right]\\&=n_\nu\xi_{\nu j}\xi_{\nu i}\mathbf{v}_\nu\cdot\frac{\partial 2\,\mathbf{A}}{\partial x_j\partial x_i}-n_\nu\xi_\nu\cdot\left[(\mathbf{v}_\nu\cdot\nabla)\left[\xi_{\nu i}\frac{\partial\mathbf{A}}{\partial x_i}\right]\right]+n_\nu\xi_\nu\left[(\mathbf{v}_\nu\cdot\nabla\xi_{\nu i})\frac{\partial\mathbf{A}}{\partial x_i}\right]\\&=n_\nu\mathbf{v}_\nu\cdot[\xi_\nu\xi_\nu:\nabla\nabla\,\mathbf{A}]-n_\nu\xi_\nu\cdot\{(\mathbf{v}_\nu\cdot\nabla)[(\xi_\nu\cdot\nabla)\mathbf{A}]\}+n_\nu\xi_\nu\cdot\left[(\mathbf{v}_\nu\cdot\nabla\xi_{\nu i})\frac{\partial\mathbf{A}}{\partial x_i}\right]\ ,\end{aligned} \tag{B7}$$

$$n_\nu\xi_\nu\cdot\left[(\mathbf{v}_\nu\cdot\nabla\xi_{\nu i})\frac{\partial \mathbf{A}}{\partial x_i}\right]=n_\nu\xi_\nu\cdot\left[\{[(\mathbf{v}_\nu\cdot\nabla)\xi_\nu]\cdot\mathbf{e}_i\}\frac{\partial \mathbf{A}}{\partial x_i}\right]$$

$$=n_\nu\xi_\nu\cdot\left\{-[(\mathbf{v}_\nu\cdot\nabla)\xi_\nu]\times\left[\mathbf{e}_i\times\frac{\partial \mathbf{A}}{\partial x_i}\right]+\left[[(\mathbf{v}_\nu\cdot\nabla)\xi_\nu]\cdot\frac{\partial \mathbf{A}}{\partial x_i}\right]\mathbf{e}_i\right\}$$

$$=-n_\nu\xi_\nu\cdot\{[(\mathbf{v}_\nu\cdot\nabla)\xi_\nu]\times\mathbf{B}\}+n_\nu[(\mathbf{v}_\nu\cdot\nabla)\xi_\nu]\cdot\frac{\partial \mathbf{A}}{\partial x_i}\xi_{\nu i}$$

$$=-n_\nu\xi_\nu\cdot\{[(\mathbf{v}_\nu\cdot\nabla)\xi_\nu]\times\mathbf{B}\}+n_\nu[(\mathbf{v}_\nu\cdot\nabla)\xi_\nu]\cdot[(\xi_\nu\cdot\nabla)\mathbf{A}]\ . \quad \text{(B8)}$$

Adding Eqs. (B5), (B7), (B8), and substracting Eq. (B6) yields, after a further simple manipulation,

$$\frac{1}{2}\frac{e_\nu}{c}n_\nu\xi_\nu\xi_\nu{:}\nabla\nabla\mathbf{A}-\frac{e_\nu}{c}n_\nu\xi_\nu\cdot\{(\mathbf{v}_\nu\cdot\nabla)[(\xi_\nu\cdot\nabla)\mathbf{A}]\}=\frac{1}{2}\frac{e_\nu}{c}\nabla\cdot([\xi_\nu\cdot(\mathbf{v}_\nu\times\mathbf{B})]n_\nu\xi_\nu-\{\xi_\nu\cdot[(\xi_\nu\cdot\nabla)\mathbf{A}]\}n_\nu\mathbf{v}_\nu)$$

$$-\frac{1}{2}\frac{e_\nu}{c}n_\nu(\mathbf{v}_\nu\times\mathbf{B})\cdot[(\xi_\nu\cdot\nabla)\xi_\nu]-\frac{1}{2}\frac{e_\nu}{c}[\xi_\nu\cdot(\mathbf{v}_\nu\times\mathbf{B})]\nabla\cdot(n_\nu\xi_\nu)$$

$$+\frac{1}{2}\frac{e_\nu}{c}n_\nu\xi_\nu\cdot\{[(\mathbf{v}_\nu\cdot\nabla)\xi_\nu-(\xi_\nu\cdot\nabla)\mathbf{v}_\nu]\times\mathbf{B}\}$$

$$+\frac{1}{2}\frac{e_\nu}{c}\{\xi_\nu\cdot[(\xi_\nu\cdot\nabla)\mathbf{A}]\}\nabla\cdot(n_\nu\mathbf{v}_\nu)\ . \quad \text{(B9)}$$

The last term of this equation can be transformed as follows, taking the equation of continuity into account:

$$\frac{1}{2}\frac{e_\nu}{c}\{\xi_\nu\cdot[(\xi_\nu\cdot\nabla)\mathbf{A}]\}\nabla\cdot(n_\nu\mathbf{v}_\nu)$$

$$=-\frac{1}{2}\frac{e_\nu}{c}\{\xi_\nu\cdot[(\xi_\nu\cdot\nabla)\mathbf{A}]\}\dot{n}_\nu$$

$$=\frac{1}{2}\frac{e_\nu}{c}\frac{\partial}{\partial t}\{n_\nu\xi_\nu\cdot[(\xi_\nu\cdot\nabla)\mathbf{A}]\}-\frac{e_\nu}{c}\left[\frac{\partial}{\partial t}[n_\nu\xi_\nu]\right]\cdot(\xi_\nu\cdot\nabla)\mathbf{A}+\frac{1}{2}\frac{e_\nu}{c}n_\nu\{\dot{\xi}_\nu\cdot[(\xi_\nu\cdot\nabla)\mathbf{A}]-\xi_\nu\cdot[(\dot{\xi}_\nu\cdot\nabla)\mathbf{A}]\}$$

$$=\frac{1}{2}\frac{e_\nu}{c}\frac{\partial}{\partial t}\{n_\nu\xi_\nu\cdot[(\xi_\nu\cdot\nabla)\mathbf{A}]\}-\frac{e_\nu}{c}\left[\frac{\partial}{\partial t}[n_\nu\xi_\nu]\right]\cdot(\xi_\nu\cdot\nabla)\mathbf{A}+\frac{1}{2}\frac{e_\nu}{c}n_\nu\{\dot{\xi}_\nu\cdot[(\xi_\nu\cdot\nabla)\mathbf{A}]-(\dot{\xi}_\nu\cdot\nabla)[\xi_{\nu,\mathrm{const}}\cdot\mathbf{A}]\}$$

$$=\frac{1}{2}\frac{e_\nu}{c}\frac{\partial}{\partial t}\{n_\nu\xi_\nu\cdot[(\xi_\nu\cdot\nabla)\mathbf{A}]\}-\frac{e_\nu}{c}\left[\frac{\partial}{\partial t}[n_\nu\xi_\nu]\right](\xi_\nu\cdot\nabla)\mathbf{A}-\frac{1}{2}\frac{e_\nu}{c}n_\nu\dot{\xi}_\nu\cdot(\xi_\nu\times\mathbf{B})\ . \quad \text{(B10)}$$

The expression for the wave energy of the cases considered ($\mathbf{v}_i=0$, $\nabla p_i=0$, $\mathbf{E}=0$) can be put in a convenient form by using the following relations. If one introduces the current density $\mathbf{j}$,

$$\mathbf{j}=e_e n_e \mathbf{v}_e\ , \quad \text{(B11)}$$

then

$$\nabla\cdot\mathbf{j}=0 \quad \text{(B12)}$$

follows from the equation of continuity for the equilibrium electrons, $\nabla\cdot(n_e\mathbf{v}_e)=-\dot{n}_e=0$. Making use of the equilibrium equation

$$\nabla p_e=\frac{1}{c}\mathbf{j}\times\mathbf{B}\ , \quad \text{(B13)}$$

one derives

$$\frac{1}{n_e}(\xi_e\cdot\nabla p_e)\nabla\cdot(n_e\xi_e)-\frac{e_\nu}{c}n_e\xi_e\cdot\{[(\mathbf{v}_e\cdot\nabla)\xi_e-(\xi_e\cdot\nabla)\mathbf{v}_e]\times\mathbf{B}\}$$

$$=\frac{1}{c}[\xi_e\cdot(\mathbf{j}\times\mathbf{B})]\nabla\cdot\xi_e+\frac{1}{cn_e}[\xi_e\cdot(\mathbf{j}\times\mathbf{B})](\xi_e\cdot\nabla n_e)-\frac{1}{c}\xi_e\cdot\left[(\mathbf{j}\cdot\nabla)\xi_e-(\xi_e\cdot\nabla)\mathbf{j}+\frac{1}{n_e}(\xi_e\cdot\nabla n_e)\mathbf{j}\right]\times\mathbf{B}$$

$$=\frac{1}{c}(\xi_e\times\mathbf{B})\cdot[-\mathbf{j}\nabla\cdot\xi_e+(\mathbf{j}\cdot\nabla)\xi_e-(\xi_e\cdot\nabla)\mathbf{j}]$$

$$=\frac{1}{c}(\xi_e\times\mathbf{B})\cdot\nabla\times(\xi_e\times\mathbf{j})$$

$$=\frac{1}{c}\nabla\cdot[(\xi_e\times\mathbf{j})\times(\xi_e\times\mathbf{B})]+\frac{1}{c}(\xi_e\times\mathbf{j})\cdot\nabla\times(\xi_e\times\mathbf{B}) \tag{B14}$$

and

$$\frac{1}{c}(\xi_e\times\mathbf{j})\cdot\nabla\times(\xi_e\times\mathbf{B})=\frac{1}{c}\xi_e\cdot[\mathbf{j}\times\nabla\times(\xi_e\times\mathbf{B})]$$

$$=\frac{1}{c}\xi_e\cdot\{\mathbf{j}\times[(\mathbf{B}\cdot\nabla)\xi_e-(\xi_e\cdot\nabla)\mathbf{B}]-(\mathbf{j}\times\mathbf{B})\nabla\cdot\xi_e\}$$

$$=\frac{1}{c}\xi_e\cdot\{\mathbf{j}\times[(\mathbf{B}\cdot\nabla)\xi_e-(\xi_e\cdot\nabla)\mathbf{B}]\}-(\xi_e\cdot\nabla p_e)\nabla\cdot\xi_e\ . \tag{B15}$$

[1] B. D. Scott, Phys. Rev. Lett. **65**, 3289 (1990).

[2] B. D. Scott, Phys. Fluids B **4**, 2468 (1992).

[3] T. M. Cherry, Trans. Cambridge Philos. Soc. **23**, 199 (1925); see E. T. Whittaker, *Analytical Dynamics* (Cambridge, London, 1937), Sec. 182, p. 142; and A. Wintner, *The Analytical Foundations of Celestial Mechanics* (Princeton University Press, Princeton, 1947), Sec. 136, p. 101.

[4] D. Pfirsch, Z. Naturforsch. Teil A **45**, 839 (1990).

[5] P. J. Morrison and M. Kotschenreuther, Reports No. DOE/ET-53088-280, IFSR 280, 1989, Institute for Fusion Studies, The University of Texas at Austin, Austin, TX 78712.

[6] M. Wakatani and A. Hasegawa, Phys. Fluids **27**, 611 (1984).

[7] P. J. Morrison and D. Pfirsch, Phys. Fluids B **2**, 1105 (1990).

[8] D. Pfirsch and P. J. Morrison, Phys. Fluids B **3**, 271 (1991).

[9] D. Pfirsch and R. N. Sudan (to be published).